A classification of unipotent spherical conjugacy classes in bad characteristic

Mauro Costantini
Dipartimento di Matematica Pura ed Applicata
Torre Archimede - via Trieste 63 - 35121 Padova - Italy
email: costantini@math.unipd.it

Abstract

Let G be a simple algebraic group over an algebraically closed field k of bad characteristic. We classify the spherical unipotent conjugacy classes of G. We also show that if the characteristic of k is 2, then the fixed point subgroup of every involutorial automorphism (involution) of G is a spherical subgroup of G.

1 Introduction

Let G be a simple algebraic group over an algebraically closed field k, with Lie algebra $\mathfrak{g}$. In this paper we determine the unipotent spherical conjugacy classes of G (we recall that a conjugacy class $\mathcal{O}$ in G is called *spherical* if a Borel subgroup of G has a dense orbit on $\mathcal{O}$) when the characteristic of k is bad for G. There has been a lot of work related to this field in the cases of good characteristic.

To fix the notation, B is a Borel subgroup of G, T a maximal torus of B, B^- the Borel subgroup opposite to B, $\{\alpha_1, \dots, \alpha_n\}$ the set of simple roots with respect to the choice of (T, B). Let W be the Weyl group of G and let us denote by s_i the reflection corresponding to the simple root α_i: $\ell(w)$ is the length of the element $w \in W$ and $\mathrm{rk}(1-w)$ is the rank of $1-w$ in the geometric representation of W.

Initially, spherical G-orbits have been studied in the context of Lie algebras ([25], [26]) in characteristic zero. The classification of spherical nilpotent orbits has been obtained by Panyushev: in terms of height, a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ is spherical if and only if its height is at most 3, which means 2 or 3 if $\mathcal{O}$ is not the zero orbit. Equivalently, $\mathcal{O}$ is spherical if and only if it contains an element of the form $e_{\gamma_1} + \cdots + e_{\gamma_t}$, where $\gamma_1, \dots, \gamma_t$ are pairwise orthogonal simple roots

(Panyushev [25], [27]).

More recently, in [5], we put our attention to spherical conjugacy classes in G over $\mathbb{C}$. We classify all spherical conjugacy classes by means of the Bruhat decomposition: a conjugacy class $\mathcal{O} \subset G$ is spherical if and only if $\dim \mathcal{O} = \ell(w) + rk(1-w)$, where $w = w(\mathcal{O})$ is the unique element of W such that $\mathcal{O} \cap BwB$ is open dense in $\mathcal{O}$ (we observe that the classification given in [5] over the complex numbers, holds in general for characteristic zero).

In [15] the authors classify spherical nilpotent orbits in good characteristic using Kempf-Russeau theory: the classification is the same as in the case of zero characteristic. In [8], the author obtains the classification of all spherical conjugacy classes in good, odd or zero characteristic by means of the Bruhat decomposition and by exploiting another characterization of spherical conjugacy classes available in good odd or zero characteristic, namely a conjugacy class $\mathcal{O}$ is spherical if and only if $\{y \in W \mid \mathcal{O} \cap ByB \neq \emptyset\} \subseteq \{w \in W \mid w^2 = 1\}$ ([6], Theorem 2.7, [7], Theorem 5.7).

In the present paper we complete the picture for unipotent spherical conjugacy classes by considering bad characteristic. Our strategy is to exhibit for each group G a set $\mathcal{O}(G)$ of unipotent conjugacy classes, show that each element in $\mathcal{O}(G)$ is spherical, and finally show that each conjugacy class not in $\mathcal{O}(G)$ is not spherical. It turns out that in bad characteristic the classification of spherical unipotent conjugacy classes is the "same" as in zero characteristic, unless $p = 2$ in type C_n and F_4, or $p = 3$ in type G_2. In these cases there are more classes than in characteristic zero. In particular, if $p = 2$ then the conjugacy class of a non-trivial unipotent element u is spherical if and only if u is an involution.

It is well known that in zero or odd characteristic the fixed point subgroup H of any involutory automorphism σ of G is a spherical subgroup (i.e. G/H is a spherical homogeneous space). This was proved by Vust in [33] in characteristic zero (see also [23] over $\mathbb{C}$). Then Springer extended the result to odd characteristic in [30]. In [28], Seitz gives an alternative proof of Springer's result. Here we prove that the result also holds in characteristic 2.

The paper is structured as follows. In Section 2 we introduce the notation. In Section 3 we recall some basic facts about the classification of unipotent conjugacy classes in bad characteristic and determine the spherical ones. We also give the list of all spherical unipotent conjugacy classes $\mathcal{O}$ for which there is an element u in $\mathcal{O}$ of the form $u = x_{\gamma_1}(1) \cdots x_{\gamma_t}(1)$, where $\gamma_1, \ldots, \gamma_t$ are

pairwise orthogonal simple roots.

In Section 4 we prove that if G is a reductive connected algebraic group in characteristic 2, and σ is any involutory automorphism of G, then the fixed point subgroup H of σ is a spherical subgroup of G.

2 Preliminaries

We denote by $\mathbb{C}$ the complex numbers, by $\mathbb{R}$ the reals, by $\mathbb{Z}$ the integers and by $\mathbb{N}$ the natural numbers.

Let $A = (a_{ij})$ be a finite indecomposable Cartan matrix of rank n with associated root system Φ, and let k be an algebraically closed field of characteristic $\operatorname{char} k = p$. Let G be a simple algebraic group over k associated to A, with Lie algebra $\mathfrak{g}$. We fix a maximal torus T of G, and a Borel subgroup B containing T: B^- is the Borel subgroup opposite to B, U (respectively U^-) is the unipotent radical of B (respectively of B^-). We denote by $\mathfrak{h}$ the Lie algebra of T. Then Φ is the set of roots relative to T, and B determines the set of positive roots Φ^+, and the simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. We fix a total ordering on Φ^+ compatible with the height function. We shall use the numbering and the description of the simple roots in terms of the canonical basis $(e_1, \ldots, e_k)$ of an appropriate $\mathbb{R}^k$ as in [3], Planches I-IX. For the exceptional groups, we shall write $\beta = (m_1, \ldots, m_n)$ for $\beta = m_1\alpha_1 + \ldots + m_n\alpha_n$. We denote by P the weight lattice, by P^+ the monoid of dominant weights and by W the Weyl group; s_i is the simple reflection associated to α_i, $\{\omega_1, \ldots, \omega_n\}$ are the fundamental weights, w_0 is the longest element of W. The real space $E = \mathbb{R}P$ is a Euclidean space, endowed with the scalar product $(\alpha_i, \alpha_j) = d_i a_{ij}$. Here $\{d_1, \ldots, d_n\}$ are relatively prime positive integers such that if D is the diagonal matrix with entries $d_1, \ldots, d_n$, then DA is symmetric.

We put $\Pi = \{1, \ldots, n\}$ and we fix a Chevalley basis $\{h_i, i \in \Pi; e_\alpha, \alpha \in \Phi\}$ of $\mathfrak{g}$.

We use the notation $x_\alpha(\xi)$, $h_\alpha(z)$, for $\alpha \in \Phi$, $\xi \in k$, $z \in k^*$ as in [32], [12]. For $\alpha \in \Phi$ we put $X_\alpha = \{x_\alpha(\xi) \mid \xi \in k\}$, the root-subgroup corresponding to α, and $H_\alpha = \{h_\alpha(z) \mid z \in k^*\}$. We identify W with N/T, where N is the normalizer of T: given an element $w \in W$ we shall denote a representative of w in N by $\dot{w}$. We choose the x_α's so that, for all $\alpha \in \Phi$,

$n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$ lies in N and has image the reflection s_α in W. Then

$$x_\alpha(\xi)x_{-\alpha}(-\xi^{-1})x_\alpha(\xi) = h_\alpha(\xi)n_\alpha \quad , \quad n_\alpha^2 = h_\alpha(-1) \tag{2.1}$$

for every $\xi \in k^*$, $\alpha \in \Phi$ ([31], Proposition 11.2.1).

For algebraic groups we use the notation in [17], [13]. In particular, for $J \subseteq \Pi$, $\Delta_J = \{\alpha_j \mid j \in J\}$, Φ_J is the corresponding root system, W_J the Weyl group, P_J the standard parabolic subgroup of G, $L_J = T\langle X_\alpha \mid \alpha \in \Phi_J\rangle$ the standard Levi subgroup of P_J. For $z \in W$ we put $U_z = U \cap z^{-1}U^- z$. Then the unipotent radical $R_u P_J$ of P_J is $U_{w_0 w_J}$, where w_J is the longest element of W_J. Moreover $U \cap L_J = U_{w_J}$ is a maximal unipotent subgroup of L_J.

If Ψ is a subsystem of type X_r of Φ and H is the subgroup generated by X_α, $\alpha \in \Psi$, we say that H is a X_r-subgroup of G.

If X is G-variety and $x \in X$, we denote by $G.x$ the G-orbit of x and by G_x the isotropy subgroup of x in G. If the homogeneous space G/H is spherical, we say that H is a spherical subgroup of G.

If x is an element of a group K and $H \leq K$, we shall also denote by $C(x)$ the centralizer of x in K, and by $C_H(x)$ the centralizer of x in H. If x, $y \in K$, then $x \sim y$ means that x, y are conjugate in K. For unipotent classes in exceptional groups we use the notation in [13]. We use the description of centralizers of involutions as in [18], [2].

We denote by $\mathbb{Z}_r$ the cyclic group of order r.

For each conjugacy class $\mathcal{O}$ in G, $w = w(\mathcal{O})$ is the unique element of W such that $BwB \cap \mathcal{O}$ is open dense in $\mathcal{O}$.

3 The classification

We recall that the bad primes for the individual types of simple groups are as follows:

none when G has type A_n;

$p = 2$ when G has type B_n, C_n, D_n;

$p = 2$ or 3 when G has type G_2, F_4, E_6, E_7;

$p = 2$, 3 or 5 when G has type E_8.

One may find a detailed account of the classification of both unipoten classes and nilpotent orbits in bad characteristic in [13], §5.11.

To deal with the classical groups with $p = 2$, we recall that the unipotent classes were determined by Wall in [34] (the nilpotent orbits were determined by Hesselink in [16]). For convenience of the reader, here we recall the classification of unipotent classes in the classical groups following [29], §2. Suppose $G = GL(n)$ (any characteristic), u a unipotent element of G. Then one can associate to u a partition $\lambda = (\lambda_1, \lambda_2, \ldots) = 1^{c(1)} \oplus 2^{c(2)} \oplus \cdots$ of n with $\lambda_1 \geq \lambda_2 \geq \cdots$, where $c(i)$ is the number of Jordan blocks of u of dimension i, for every $i \geq 1$. In this way the set $\mathcal{CU}(G)$ of unipotent conjugacy classes of G is parametrized by the set of partitions of n. We denote by C_λ the unipotent class corresponding to the partition λ. The set $\mathcal{CU}(G)$ has a natural partial order: $\mathcal{O}_1 \leq \mathcal{O} \Leftrightarrow \mathcal{O}_1 \subseteq \overline{\mathcal{O}}$. If we partially order the set of partitions of n by $\lambda \leq \mu \Leftrightarrow \sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j$ for every $i \geq 1$, then the map $\lambda \to C_\lambda$ is an isomorphism of p.o.-sets.

Now assume $p = 2$. In this case there exists a homomorphism (central isogeny) of $SO(2n+1)$ onto $Sp(2n)$ which is an isomorphism of abstract groups. We shall therefore deal only with $Sp(2n)$ and $SO(2n)$. Let ω be an object distinct from 0 and 1, and consider the set $\{\omega, 0, 1\}$ totally ordered by $\omega < 0 < 1$. Assume $G = Sp(2n) \leq GL(2n)$ (resp. $G = O(2n) \leq Sp(2n)$). The unipotent conjugacy classes of G are parametrized by pairs (λ, ε) such that

a) $\lambda = 1^{c(1)} \oplus 2^{c(2)} \oplus \cdots$ is a partition of $2n$ with $c(i)$ even for every odd i.

b) $\varepsilon : \mathbb{N} \to \{\omega, 0, 1\}$ is such that

   b$_1$) $\varepsilon(i) = \omega$ if i is odd or $i \geq 1$ and $c(i) = 0$.

   b$_2$) $\varepsilon(i) = 1$ if i is even and $c(i)$ is odd ($i \neq 0$).

   b$_3$) $\varepsilon(i) \neq \omega$ if i is even and $c(i) \neq 0$ ($i \neq 0$).

   b$_4$) $\varepsilon(0) = 1$ (resp. $\varepsilon(0) = 0$).

The correspondence is obtained as follows. Let u be a unipotent element of G. Then u determines a class in $GL(2n)$, hence the partition λ of $2n$. This partition satisfies a). Now, if i is even, $i \neq 0$ and $c(i) \neq 0$, we put $\varepsilon(i) = 0$ if $f((u-1)^{i-1}(x), x) = 0$ for every $x \in \ker(u-1)^i$, and $\varepsilon(i) = 1$ otherwise (here f is the bilinear form used to define $Sp(2n)$). In view of condition b), this defines uniquely ε.

We denote by $C_{\lambda,\varepsilon}$ the unipotent class of G corresponding to (λ, ε). We observe that every unipotent class of $Sp(2n)$ intersects $O(2n)$ in a unique class of $O(2n)$. Moreover, the unipotent classes of $O(2n)$ contained in $SO(2n)$ (the connected component of $O(2n)$) are those for which λ_1^* is even (λ^* is the dual partition of λ). If all λ_i's and $c(i)$'s are even and if $\varepsilon(i) \neq 1$ for every i, then the unipotent class $C_{\lambda,\varepsilon}$ of $O(2n)$ splits into two classes of $SO(2n)$. All the other unipotent classes in $SO(2n)$ are unipotent classes in $O(2n)$.

We shall use the notation $(\lambda, \varepsilon) = 1_{\varepsilon(1)}^{c(1)} \oplus 2_{\varepsilon(2)}^{c(2)} \oplus \cdots$.

In [29], §2.8, 2.9, 2.10, there are formulas for the dimensions of centralizers of unipotent elements in $Sp(2n)$, $O(2n)$ (hence also in $SO(2n)$), the determination of the component groups $C(u)/C(u)^\circ$ in the various cases, and an explicit definition of a partial order on pairs (λ, ε) such that $C_{\lambda,\varepsilon} \leq C_{\mu,\phi} \Leftrightarrow (\lambda, \varepsilon) \leq (\mu, \phi)$.

We shall use the notation as in [29]. As above mentioned, for the classical groups we only have to consider $p = 2$, and then groups of type C_n and D_n. For convenience, we shall work with $Sp(2n)$ and $SO(2n)$.

Strategy of the proof. Let $G_{\mathbb{C}}$ be the corresponding group over $\mathbb{C}$. We have shown in [5] that for every spherical conjugacy class $\mathcal{C}$ of $G_{\mathbb{C}}$ there exists an involution $w = w(\mathcal{C})$ in W such that $\dim \mathcal{C} = \ell(w) + \mathrm{rk}(1-w)$, with $\mathcal{C} \cap BwB \neq \emptyset$. For each group G we introduce a certain set $\mathcal{O}(G)$ of unipotent conjugacy classes which are candidates for being spherical. For each $\mathcal{O} \in \mathcal{O}(G)$ we show that there is a (non-necessarily unipotent) spherical conjugacy class $\mathcal{C}$ in $G_{\mathbb{C}}$ such that

$$\dim \mathcal{O} = \dim \mathcal{C}$$

Let $w = w(\mathcal{C})$. Our aim is to show that $\mathcal{O} \cap BwB \neq \emptyset$.

Definition 3.1 *Let $\mathcal{O}$ be a conjugacy class of G. We say that $\mathcal{O}$ satisfies $(*)$ if there exists $w \in W$ such that $BwB \cap \mathcal{O} \neq \emptyset$ and $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1-w)$.*

Let $\mathcal{O}$ be a conjugacy class in G. There exists a unique element $w = w(\mathcal{O}) \in W$ such that $\mathcal{O} \cap BwB$ is open dense in $\mathcal{O}$. In particular

$$\overline{\mathcal{O}} = \overline{\mathcal{O} \cap BwB} \subseteq \overline{BwB}. \tag{3.1}$$

It follows that if y is an element of $\overline{\mathcal{O}}$ and $y \in BwB$, then $w \leq z$ in the Chevalley-Bruhat order of W.

We recall the following result proved in [5], Theorem 5 over $\mathbb{C}$, but which is valid with the same proof over any algebraically closed field.

Theorem 3.2 *Suppose that $\mathcal{O}$ contains an element $x \in BwB$. Then*

$$\dim B.x \geq \ell(w) + \mathrm{rk}(1-w).$$

In particular $\dim \mathcal{O} \geq \ell(w) + \mathrm{rk}(1-w)$. If, in addition, $\dim \mathcal{O} \leq \ell(w) + \mathrm{rk}(1-w)$ then $\mathcal{O}$ is spherical, $w = w(\mathcal{O})$ and $B.x$ is the dense B-orbit in $\mathcal{O}$. □

Let $\mathcal{O}$ be a conjugacy class of G and let $w = w(\mathcal{O})$. If $\mathcal{O}^{-1} = \mathcal{O}$ (i.e. if any element $x \in \mathcal{O}$ is conjugate to its inverse), then $w^2 = 1$. It is well known that over any algebraically closed field any unipotent element is conjugate to its inverse ([9], Lemma 1.16, [10], Lemma 2.3. See also [22], Proposition 2.5 (a)), so that w is an involution for every non-trivial unipotent conjugacy class $\mathcal{O}$. However, it has recently been shown in [14], that $w^2 = 1$ for every conjugacy class $\mathcal{O}$ in G.

If g is in $Z(G)$, then $g \in T$, $\mathcal{O}_g = \{g\}$, $w(\mathcal{O}) = 1$. In the remaining of the paper we shall consider only non-central conjugacy classes.

We shall use the following result

Lemma 3.3 *Assume the positive roots $\beta_1, \ldots, \beta_\ell$ are such that $[X_{\pm\beta_i}, X_{\pm\beta_j}] = 1$ for every $1 \leq i < j \leq \ell$. Then, for $\xi_1, \ldots, \xi_\ell \in k^*$, $g = x_{\beta_1}(-\xi_1^{-1}) \cdots x_{\beta_\ell}(-\xi_\ell^{-1})$, $h = h_{\beta_1}(-\xi_1) \cdots h_{\beta_\ell}(-\xi_\ell)$ we have*

$$g x_{-\beta_1}(\xi_1) \cdots x_{-\beta_\ell}(\xi_\ell) g^{-1} = n_{\beta_1} \cdots n_{\beta_\ell} h x_{\beta_1}(2\xi_1^{-1}) \cdots x_{\beta_\ell}(2\xi_\ell^{-1})$$

Proof. By (2.1) we have $x_\alpha(-\xi^{-1}) x_{-\alpha}(\xi) x_\alpha(\xi^{-1}) = n_\alpha h_\alpha(-\xi) x_\alpha(2\xi^{-1})$. The result follows from $[X_{\pm\beta_i}, X_{\pm\beta_j}] = 1$ for every $1 \leq i < j \leq \ell$. □

The hypothesis of the Lemma are satisfied for instance if $\beta_1, \ldots, \beta_\ell$ are pairwise orthogonal and long, as in [11], Lemma 4.1. In characteristic 2, we have $[X_\gamma, X_\delta] = 1$ for every pair (γ, δ) of orthogonal roots, so that for any set of pairwise orthogonal roots $\beta_1, \ldots, \beta_\ell$ and for $g = x_{\beta_1}(1) \cdots x_{\beta_\ell}(1)$ we get

$$g x_{-\beta_1}(1) \cdots x_{-\beta_\ell}(1) g^{-1} = n_{\beta_1} \cdots n_{\beta_\ell} \quad \text{for } p = 2. \tag{3.2}$$

3.1 Classical groups in characteristic 2.

As mentioned above, for the classical groups we have deal only with $Sp(2n)$ and $SO(2n)$. However, for completeness, we shall also deal with the case when G has type A_n, since this is dealt with in [15], but not in [8].

3.1.1 Type A_n, $n \geq 1$.

We show that every spherical unipotent conjugacy class satisfies $(*)$. The spherical nilpotent orbits (and therefore the spherical unipotent classes) have been classified in [15], and it follows that a unipotent conjugacy class $\mathcal{O}$ is spherical if and only if $\mathcal{O} = X_i$ the unipotent class $2^i \oplus 1^{n+1-2i}$ for $i = 1, \dots, m = \left[\frac{n+1}{2}\right]$. For every $i = 1, \dots, m$, let $\beta_i = e_i - e_{n+2-i}$: then, as for $\mathbb{C}$ (and for any algebraically closed field of odd or zero characteristic) the element $u = x_{-\beta_1}(1) \cdots x_{-\beta_i}(1)$ lies in $X_i \cap BwB$, where $w = s_{\beta_1} \cdots s_{\beta_i}$ is such that $\dim X_i = \ell(w) + \mathrm{rk}(1-w)$. In fact one may take $n_{\beta_1} \cdots n_{\beta_i} \in s_{\beta_1} \cdots s_{\beta_i} B \cap X_i$.

$\mathcal{O}$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$
$X_\ell = 2^\ell \oplus 1^{n+1-2\ell}$ $\ell = 1, \dots, m = \left[\frac{n+1}{2}\right]$	$s_{\beta_1} \cdots s_{\beta_\ell}$	$n_{\beta_1} \cdots n_{\beta_\ell}$

Table 1: Spherical unipotent classes in A_n ($p = 2$).

In particular, a non-trivial unipotent class $\mathcal{O}$ is spherical if and only if it consists of involutions, if and only if $\mathcal{O}$ has a representative of the form $x_{\gamma_1}(1) \cdots x_{\gamma_t}(1)$, where $\gamma_1, \dots, \gamma_t$ are pairwise orthogonal simple roots.

3.1.2 Type C_n (and B_n), $n \geq 2$.

We first show that if u is an involution of G, then $\mathcal{O}_u$ is spherical, by showing that $\mathcal{O}_u$ satisfies $(*)$. So let u be an involution of $G = Sp(2n)$. Then the partition corresponding to u is of the form $\lambda = 1^{c_1} \oplus 2^{c_2}$, with $c_2 = \ell$, $c_1 = 2n - 2\ell$,

Using the above recalled description of unipotent conjugacy classes, let $\lambda = 2^i \oplus 1^{2n-2i}$, for $i = 1, \dots, n$. Then we have $\varepsilon_0 = 1$, $\varepsilon_1 = \omega$, $\varepsilon_i = 0$ for $i \geq 3$. As for ε_2, we have $\varepsilon_2 = 1$ if i is odd. On the other hand, if i is even, we have both possibilities $\varepsilon_2 = 0$ or 1. We denote by X_i the class corresponding to $\varepsilon_2 = 1$, and by Y_i the class corresponding to $\varepsilon_2 = 0$ (when i is even)

We denote by $X_{\ell,\mathbb{C}}$ the unipotent class in $Sp(2n,\mathbb{C})$ corresponding to $\lambda = 2^\ell \oplus 1^{2n-2\ell}$. Then we get

$$\begin{array}{lclcll} \dim X_\ell & = & \dim X_{\ell,\mathbb{C}} & = & \ell(2n-\ell+1) & \ell \in \{1,\dots,n\} \\ \dim Y_\ell & = & \dim X_{\ell,\mathbb{C}} - \ell & = & 2n\ell - \ell^2 = \ell(2n-\ell) & \ell \in \{1,\dots,n\},\ \ell \text{ even} \end{array}$$

Note that if ℓ is even and we write $\ell = 2\ell'$, then $\dim Y_\ell = \dim \mathcal{O}_{\sigma_{\ell'},\mathbb{C}}$, where $\mathcal{O}_{\sigma_{\ell'},\mathbb{C}}$ is the conjugacy class in $Sp(2n,\mathbb{C})$ of the involution $\sigma_{\ell'}$ ([5], Table 1). In $Sp(2n,\mathbb{C})$, the spherical semisimple conjugacy class $\mathcal{O}_{\sigma_{\ell'}}$ lies over $w = s_{\gamma_1}\cdots s_{\gamma_{\ell'}}$ ([5], Table 5, [11], §4.2.2).

We observe that if the partition associated to the involution u is $\lambda = 2^{2\ell} \oplus 1^{2n-4\ell}$, then $\mathcal{O}_u = 2_0^{2\ell} \oplus 1^{2n-4\ell}$ if and only if $f((u-1)v,v) = 0$ for every $v \in V$ (here f is the bilinear form on V used to define $Sp_{2n}(k)$). Let w be an involution of W, $L(w) = \{\beta \in \Phi^+ \mid w(\beta) = -\beta, \beta \text{ long}\}$, $L(w)_\perp = \{\gamma \in \Phi^+ \mid w(\gamma) = -\gamma, (\gamma, L(w)) = 0, \gamma \text{ short}\}$. Then $w = \prod_{\beta\in L(w)} s_\beta \prod_{\gamma\in L(w)_\perp} s_\gamma$. Let $x = \prod_{\beta\in L(w)} n_\beta \prod_{\gamma\in L(w)_\perp} n_\gamma$. Then x is an involution in BwB and the number of blocks of length 2 in the Jordan canonical form of x is $|\,L(w)\,| + 2|\,L(w)_\perp\,|$. If this number is even, then $f((x-1)v,v) = 0$ for every $v \in V$ if and only if $L(w) = \emptyset$.

We put $\beta_i = 2e_i$ for $i = 1,\dots,n$ and $\gamma_i = e_{2i-1} + e_{2i}$ for $\ell = 1,\dots,p = [\frac{n}{2}]$.

Then it is straightforward to show that

$$x_{-\beta_1}(1)\cdots x_{-\beta_\ell}(1) \in X_\ell \cap Bs_{\beta_1}\cdots s_{\beta_\ell}B \cap U^- \quad \text{for } \ell = 1,\dots,n$$

$$x_{-\gamma_1}(1)\cdots x_{-\gamma_\ell}(1) \in Y_{2\ell} \cap Bs_{\gamma_1}\cdots s_{\gamma_\ell}B \cap U^- \quad \text{for } \ell = 1,\dots,p$$

By (3.2), we can choose

$$n_{\beta_1}\cdots n_{\beta_\ell} \in X_\ell \cap wB \quad \text{for } \ell = 1,\dots,n.$$

$$n_{\gamma_1}\cdots n_{\gamma_\ell} \in Y_{2\ell} \cap wB \quad \text{for } \ell = 1,\dots,p.$$

One can easily deduce which classes of involutions have a representative of the form $u = \prod_{i\in K} x_{\alpha_i}(1)$ for a certain subset K of Π. Note that since u is an involution, then $(\alpha_i,\alpha_j) = 0$ if i, $j \in K$ with $i \neq j$. Up to the W action, we have only the following subsets K, and the corresponding classes:

$$\prod_{i=1}^{\ell} x_{\alpha_{n-2(i-1)}}(1) \in X_{2\ell-1} \quad \text{for } \ell = 1,\dots,\left[\frac{n+1}{2}\right]$$

$$\prod_{i=1}^{\ell} x_{\alpha_{2i-1}}(1) \in Y_{2\ell} \quad \text{for } \ell = 1, \ldots, p = \left[\frac{n}{2}\right]$$

These exhaust the conjugacy classes of involutions with representative of the form $\prod_{i\in K} x_{\alpha_i}(1)$. In particular all X_{2i} have no representative of the form $\prod_{i\in K} x_{\alpha_i}(1)$. The point is that in good characteristic, for $\ell = 1, \ldots, p = \left[\frac{n}{2}\right]$ the element $\prod_{i=1}^{\ell} x_{\alpha_{2i-1}}(1)$ is conjugate to $\prod_{i=1}^{2\ell} x_{\beta_i}(1)$ (which lies in $X_{2\ell}$).

If $J_i = \{i+1, \ldots, n\}$ ($J_n = \varnothing$) for $i = 1, \ldots, n$, $K_\ell = \{1, 3, \ldots, 2\ell-1, 2\ell+1, 2\ell+2, \ldots, n\}$ for $\ell = 1, \ldots, p$, we obtained

$\mathcal{O}$	J	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$X_\ell = 2^\ell \oplus 1^{2n-2\ell}$ $\ell = 1, \ldots, n$	J_ℓ	$s_{\beta_1} \cdots s_{\beta_\ell}$	$n_{\beta_1} \cdots n_{\beta_\ell}$	$\ell(2n-\ell+1)$
$Y_{2\ell} = 2_0^{2\ell} \oplus 1^{2n-4\ell}$ $\ell = 1, \ldots, p = [\frac{n}{2}]$	K_ℓ	$s_{\gamma_1} \cdots s_{\gamma_\ell}$	$n_{\gamma_1} \cdots n_{\gamma_\ell}$	$4\ell(n-\ell)$

Table 2: Involutions in C_n, $n \geq 2$, $p = 2$.

where $w(\mathcal{O}) = w_0 w_J$. By Theorem 3.2, we have proved

Proposition 3.4 *Let $\mathcal{O}$ be the conjugacy class of an involution of $Sp(2n)$ in characteristic 2. Then $\mathcal{O}$ is spherical.* $\square$

Our aim is to show that a (non-trivial) unipotent conjugacy class $\mathcal{O}_u$ is spherical if and only if u is an involution. By [15], Remark 2.14, the orbit $\mathcal{O}$ is spherical if and only if G/H is spherical, where H is the isotropy subgroup of an element in $\mathcal{O}$. Moreover G/H is spherical if and only if G/H° is spherical, where H° denotes the connected component of H. We shall therefore use the following

Lemma 3.5 *Let $\mathcal{O}$ be a G-orbit with isotropy subgroup H. Then $\mathcal{O}$ is spherical if and only if G/H° is spherical.* $\square$

By Proposition 3.4, we are left to show that if the (non-trivial) unipotent class $\mathcal{O}$ does not consist of involutions, then $\mathcal{O}$ is not spherical. Let u be a unipotent element of order greater than 4, and let v be an element of order 4 in the subgroup generated by u. Since $C(u) \leq C(v)$, if

$\mathcal{O}_v$ is non-spherical, then also $\mathcal{O}_u$ is non-spherical. We are therefore left to consider the set X of conjugacy classes of elements of order 4. By [19], Theorem 2.2, it is enough to show that the minimal elements in X are not spherical.

From the explicit definition of a partial order on pairs (λ, ε) such that $C_{\lambda,\varepsilon} \leq C_{\mu,\phi} \Leftrightarrow (\lambda, \varepsilon) \leq (\mu, \phi)$ given in [29], §2.10, it follows that the minimal elements in X are the classes $3^2 \oplus 1^{2n-6}$ (if $n \geq 3$) and $4 \oplus 1^{2n-4}$.

In the following lemma we deal with these cases. We also consider a case in D_n.

Lemma 3.6 *Let $\mathcal{O}$ be the unipotent conjugacy class of type $3^2 \oplus 1^{2n-6}$ in C_n or D_n. Then $\mathcal{O}$ is not spherical (*$\mathrm{char}\, k = 2$*).*

Proof. Let u be an element in $\mathcal{O} = 3^2 \oplus 1^{2n-6}$ (this exists in C_n if $n \geq 3$). In $Sp(2n)$, we may take $u = x_{\alpha_2}(1)x_{\gamma-\alpha_2}(1)$, where γ is the highest short root ($\gamma = e_1 + e_2$), and get $C(u)^\circ \leq P$ where P is the maximal parabolic subgroup $P_{I\setminus\{\alpha_2\}}$. Then $C(u)^\circ = CR$, $C = H_{\alpha_1} \times K$, where K is the C_{n-3}-subgroup of G corresponding to $\{\alpha_4, \dots, \alpha_n\}$, and R is the subgroup

$$\{u \in R_uP \mid u = \prod_{\alpha\in\Phi^+} x_\alpha(z_\alpha) \mid z_{a_2} = z_{\gamma-\alpha_2}\}$$

of codimension 1 in R_uP. It follows that $C(u)^\circ$ fixes the element $e_{\alpha_2} + e_{\gamma-\alpha_2}$ of $\mathfrak{g}$. However, we clearly have $C(u) \leq C(u^2)$, and $u^2 = x_\gamma(1)$. But $C(x_\gamma(1))$ fixes the element e_γ of $\mathfrak{g}$, since $x_\gamma(1) = 1 + e_\gamma$ in $M_{2n}(k)$. It follows that $C(u)^\circ$ has 2 linearly independent invariants in $\mathfrak{g}$, so that $Sp(2n)/C(u)^\circ$ is not spherical.

Since u lies in $SO(2n)$, and both $e_{\alpha_2} + e_{\gamma-\alpha_2}$ and e_γ are in the Lie algebra of $SO(2n)$, the $SO(2n)$-orbit of u is not spherical as well. □

Lemma 3.7 *Let $\mathcal{O}$ be the unipotent conjugacy class of type $4 \oplus 1^{2n-4}$ in C_n. Then $\mathcal{O}$ is not spherical (*$\mathrm{char}\, k = 2$*).*

Proof. Let u be an element in $\mathcal{O} = 4 \oplus 1^{2n-4}$. In $Sp(2n)$, we may take $u = x_{\alpha_1}(1)x_\delta(1)$, where $\delta = 2e_2$, and get $C(u)^\circ \leq P$, where P is the parabolic subgroup $P_{I\setminus\{\alpha_1,\alpha_2\}}$. Then $C(u)^\circ = CR$, where C is the C_{n-2}-subgroup of G corresponding to $\{\alpha_3, \dots, \alpha_n\}$, and R is a subgroup of U. In fact $\dim R = 2n - 2$, and R is the product of X_α's, where $\alpha = e_1 \pm e_i$, $i = 3, \dots, n$ or $\alpha = e_1 + e_2$ or $\alpha = 2e_1$.

It follows that $C(u)^\circ$ fixes the first 2 basis vectors v_1 and v_2 of the natural module of G, so that $Sp(2n)/C(u)^\circ$ is not spherical. $\square$

We have proved

Proposition 3.8 *Let $\mathcal{O}$ be a non-trivial unipotent conjugacy class of $Sp(2n)$ in characteristic 2. Then $\mathcal{O}$ is spherical if and only if it consists of involutions.* $\square$

3.1.3 Type D_n, $n \geq 4$.

Let $m = \left[\frac{n}{2}\right]$. We put $\beta_i = e_{2i-1} + e_{2i}$, $\delta_i = e_{2i-1} - e_{2i}$ for $i = 1, \ldots, m$. For $\ell = 1, \ldots, m-1$ we put $J_\ell = \{2\ell+1, \ldots, n\}$, $J_m = \varnothing$, $K_\ell = J_\ell \cup \{1, 3, \ldots, 2\ell-1\}$ for $\ell = 1, \ldots, m$. Also we put $K'_m = \{1, 3, \ldots, n-3, n\}$.

Let u be an involution of $G = SO(2n)$. Then the partition corresponding to u is of the form $\lambda = 2^{c_2} \oplus 1^{c_1}$, with $c_2 = 2i$, $c_1 = 2n - 4i$, $i = 1, \ldots, m$.

For each $\ell = 1, \ldots, m-1$ there are 2 conjugacy classes corresponding to $\lambda = 2^{2i} \oplus 1^{2n-4i}$: we denote by X_ℓ the class $2_0^{2\ell} \oplus 1^{2n-4\ell}$, and by Z_ℓ the class $2^{2\ell} \oplus 1^{2n-4\ell}$. If $\ell = m$, then we denote by Z_m the class $2^{2m} \oplus 1^{2n-4m}$. The conjugacy class in $O(2n)$ corresponding to $2_0^{2m} \oplus 1^{2n-4m}$ is a single class X_m in $SO(2n)$ if n is odd, while it splits into 2 conjugacy classes X_m and X'_m in $SO(2n)$ if n is even.

We have

$$\dim Z_\ell = 4\ell(n-\ell) \quad , \quad \dim X_\ell = 2\ell(2n-2\ell-1) \quad \text{for } \ell = 1, \ldots, m$$

with $\dim X'_m = \dim X_m$ if n is even.

We have chosen the notation so that for each conjugacy class of involutions $\mathcal{O}$ in G, the conjugacy class $\mathcal{C}$ in $G_{\mathbb{C}}$ denoted by the same symbol in [11] §4.3, has the same dimension. For the corresponding w, we write w as a product of commuting reflections, $w = s_{\gamma_1} \cdots s_{\gamma_t}$. It is straightforward to prove that in each case the element $x = n_{\gamma_1} \cdots n_{\gamma_t}$ lies in $\mathcal{O}$. We summarize in the following tables the results obtained:

$\mathcal{O}$	J	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$Z_\ell = 2^{2\ell} \oplus 1^{2n-4\ell}$ $\ell = 1, \ldots, m$	J_ℓ	$s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell}$	$n_{\beta_1}n_{\delta_1}\cdots n_{\beta_\ell}n_{\delta_\ell}$	$4\ell(n-\ell)$
$X_\ell = 2_0^{2\ell} \oplus 1^{2n-4\ell}$ $\ell = 1, \ldots, m$	K_ℓ	$s_{\beta_1}\cdots s_{\beta_\ell}$	$n_{\beta_1}\cdots n_{\beta_\ell}$	$2\ell(2n-2\ell-1)$
$X'_m = (2_0^{2m})'$	K'_m	$s_{\beta_1}\cdots s_{\beta_{m-1}}s_{\alpha_{n-1}}$	$n_{\beta_1}\cdots n_{\beta_{m-1}}n_{\alpha_{n-1}}$	$n(n-1)$

Table 3: Involutions in D_n, $n \geq 4$, $n = 2m$.

$\mathcal{O}$	J	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$Z_\ell = 2^{2\ell} \oplus 1^{2n-4\ell}$ $\ell = 1, \ldots, m$	J_ℓ	$s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell}$	$n_{\beta_1}n_{\delta_1}\cdots n_{\beta_\ell}n_{\delta_\ell}$	$4\ell(n-\ell)$
$X_\ell = 2_0^{2\ell} \oplus 1^{2n-4\ell}$ $\ell = 1, \ldots, m$	K_ℓ	$s_{\beta_1}\cdots s_{\beta_\ell}$	$n_{\beta_1}\cdots n_{\beta_\ell}$	$2\ell(2n-2\ell-1)$

Table 4: Involutions in D_n, $n \geq 4$, $n = 2m+1$.

By Theorem 3.2, we have proved

Proposition 3.9 *Let $\mathcal{O}$ be the conjugacy class of an involution of $SO(2n)$ in characteristic 2. Then $\mathcal{O}$ is spherical.* □

Our aim is to show that a (non-trivial) unipotent conjugacy class $\mathcal{O}_u$ is spherical if and only if u is an involution. Using the same arguments as in case C_n, we are left to consider the set X of conjugacy classes of elements of order 4, and then show that the minimal elements in X are not spherical. From the explicit definition of a partial order on pairs (λ, ε) such that $C_{\lambda,\varepsilon} \leq C_{\mu,\phi} \Leftrightarrow (\lambda, \varepsilon) \leq (\mu, \phi)$ given in [29], §2.10, it follows that the minimal element in X is the class $3^2 \oplus 1^{2n-6}$. By Lemma 3.6, this class is not spherical. We have therefore proved

Proposition 3.10 *Let $\mathcal{O}$ be a non-trivial unipotent conjugacy class of $SO(2n)$ in characteristic 2. Then $\mathcal{O}$ is spherical if and only if it consists of involutions.* □

Remark 3.11 From our discussion, it follows that for D_n the map $\pi_G : X^{G'} \to X^G$ defined in [29], Theorem III.5.2 induces an isomorphism of p.o. sets between $X^{G'}_{\text{sph}} \to X^G_{\text{sph}}$ where $X^{G'}_{\text{sph}}$, X^G_{sph} are the corresponding sets of spherical unipotent classes. In particular, every spherical unipotent conjugacy class has a representative of the form $\prod_{\alpha \in K} x_\alpha(1)$ for a certain set of pairwise orthogonal simple roots K.

3.2 Exceptional groups.

For the exceptional groups, we use [21], Table 2. In this table, for each group G, there are all unipotent conjugacy classes $\mathcal{O}_u$, in every characteristic, for which the dimension of $C(u)$ is greater than a certain number l_G. From this we deduce the following table

G	$\dim B$	u with $\dim \mathcal{O}_u \leq \dim B$	$\dim \mathcal{O}_u$	$\lvert C(u)/C(u)^\circ \rvert$
E_6	42	$A_1,\ 2A_1,\ 3A_1,\ A_2$	$22, 32, 40, 42$	$1, 1, 1, 2$
E_7	70	$A_1,\ 2A_1,\ 3A_1'',\ 3A_1',\ A_2,\ 4A_1$	$34, 52, 54, 64, 66, 70$	$1, 1, 1, 1, 2, 1$
E_8	128	$A_1,\ 2A_1,\ 3A_1,\ A_2,\ 4A_1$	$58, 92, 112, 114, 128$	$1, 1, 1, 2, 1$
F_4	28	$A_1,\ \tilde{A}_1(p=2),\ \tilde{A}_1(p \neq 2),\ \tilde{A}_1^{(2)}(p=2),\ A_1\tilde{A}_1$	$16, 16, 22, 22, 28$	$1, 1, 2, 1, 1$
G_2	8	$A_1,\ \tilde{A}_1(p=3),\ \tilde{A}_1(p \neq 3),\ \tilde{A}_1^{(3)}(p=3)$	$6, 6, 8, 8$	$1, 1, 1, 1$

Table 5: Unipotent classes of small dimension in exceptional groups.

where the value of $\lvert C(u)/C(u)^\circ \rvert$ for E_7 refers to the adjoint group (see [1]).

The unipotent conjugacy classes appearing in Table 5 are the only candidates to being spherical. We shall show that they are all spherical, except for the classes of type A_2 in E_6, E_7 and E_8. Note that when $p = 2$, then all classes of involutions appear in Table 5 by the results in [2].

Lemma 3.12 *Let $\mathcal{O}$ be the unipotent conjugacy class of type A_2 in E_6, E_7 or E_8. Then $\mathcal{O}$ is not spherical (in any characteristic).*

Proof. Let u be an element in $\mathcal{O}$. From [21], Table 2, it follows that the type of $C(u)^\circ$ is independent of the characteristic. For completeness, we determine $C(u)^\circ$ in all cases.

In E_8, we may take $u = x_{\alpha_8}(1)x_{\beta-\alpha_8}(1)$, where β is the highest root, and get $C(u)^\circ \leq P$ where P is the maximal parabolic subgroup $P_{I\setminus\{\alpha_8\}}$. Then $C(u)^\circ = CR$, where C is the E_6-subgroup of G corresponding to $\{\alpha_1, \dots, \alpha_6\}$, and

$$R = \{g = \prod_{\alpha \in \Phi^+} x_\alpha(k_\alpha) \in R_u P \mid k_{\alpha_8} = k_{\beta-\alpha_8}\}.$$

In E_7, we may take $u = x_{\alpha_1}(1)x_{\beta-\alpha_1}(1)$, where β is the highest root, and get $C(u)^\circ \leq P$ where P is the maximal parabolic subgroup $P_{I\setminus\{\alpha_1\}}$. Then $C(u)^\circ = CR$, where C is the A_5-subgroup of G corresponding to $\{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$, and

$$R = \{g = \prod_{\alpha \in \Phi^+} x_\alpha(k_\alpha) \in R_u P \mid k_{\alpha_1} = k_{\beta-\alpha_1}\}.$$

In E_6, we may take $u = x_{\alpha_2}(1)x_{\beta-\alpha_2}(1)$, where β is the highest root, and get $C(u)^\circ \leq P$ where P is the maximal parabolic subgroup $P_{I\setminus\{\alpha_2\}}$. Then $C(u)^\circ = CR$, where C is the $A_2\times A_2$-subgroup of G corresponding to $\{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$, and

$$R = \{g = \prod_{\alpha\in\Phi^+} x_\alpha(k_\alpha) \in R_uP \mid k_{\alpha_2} = k_{\beta-\alpha_2}\}.$$

It is well known that the class A_2 is not spherical in E_6, E_7 or E_8 over any algebraically closed field of characteristic zero. We may now apply [4], Theorem 2.2 (i). Note that the groups $C(u)^\circ$ involved are all defined over $\mathbb{Z}$, and the argument in the proof of [4], Theorem 2.2 (i) is valid in our situation. Therefore $G/C(u)^\circ$ is not spherical in any positive characteristic. It follows that $\mathcal{O}_u$ is not spherical by Lemma 3.5. □

3.2.1 Type E_6.

We put

$$\begin{array}{ll} \beta_1 = (1,2,2,3,2,1), & \beta_2 = (1,0,1,1,1,1) \\ \beta_3 = (0,0,1,1,1,0), & \beta_4 = (0,0,0,1,0,0) \end{array}$$

For groups of type E_6 we have to consider $p = 2$, 3. If $p = 3$, then we may apply the arguments in [5], Theorem 13 to prove that the orbits of type A_1, $2A_1$ and $3A_1$ satisfy $(*)$, hence are spherical, since to handle $3A_1$ we need results for D_4 which holds due to [8], Theorem 3.4 and its proof (in fact what we need is that the maximal spherical unipotent conjugacy class $\mathcal{O}'$ of D_4 satisfies $(*)$ when $p = 3$). So now assume $p = 2$. Then again we may use the proof of [5], Theorem 3.4 to deal with A_1 and $2A_1$. Note that in these cases

$$x_{-\beta_1}(1) \in A_1 \cap Bs_{\beta_1}B \cap U^- \quad , \quad x_{-\beta_1}(1)x_{-\beta_2}(1) \in 2A_1 \cap Bs_{\beta_1}s_{\beta_2}B \cap U^-$$

with $x_{-\beta_1}(1) \sim n_{\beta_1}$, $x_{-\beta_1}(1)x_{-\beta_2}(1) \sim n_{\beta_1}n_{\beta_2}$. To deal with $3A_1$, we still may use the arguments in the proof of [5], Theorem 3.4 since we have shown in §3.1.3 that the maximal spherical unipotent conjugacy class $\mathcal{O}'$ of D_4 satisfies $(*)$ when $p = 2$, or directly observe that $x = n_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\alpha_4}$ is an involution in $Bn_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\alpha_4}B = Bw_0B$. Then $\dim\mathcal{O}_x \geq \ell(w_0) + \mathrm{rk}(1-w_0) = 40$ by Theorem 3.2, so that $x \in 3A_1$ by Table 5, since elements in A_2 are not involutions. We have proved

Proposition 3.13 *Let $\mathcal{O}$ be a non-trivial unipotent conjugacy class in E_6. Then $\mathcal{O}$ is spherical if and only if $\mathcal{O} = A_1, 2A_1$ or $3A_1$. In each case $\mathcal{O}$ satisfies $(*)$. If $\operatorname{char} k = 2$, these are precisely the classes consisting of involutions.*

3.2.2 Type E_7.

We put

$$\beta_1 = (2,2,3,4,3,2,1),\ \beta_2 = (0,1,1,2,2,2,1),\ \beta_3 = (0,1,1,2,1,0,0),$$
$$\beta_4 = \alpha_7, \quad \beta_5 = \alpha_5, \quad \beta_6 = \alpha_3, \quad \beta_7 = \alpha_2$$

For groups of type E_7 we have to consider $p = 2$, 3. If $p = 3$, then we may apply the arguments in [5], Theorem 13 to prove that the orbits of type A_1, $2A_1$, $(3A_1)'$, $(3A_1)''$ and $4A_1$ are spherical, since we need results for D_n which holds due to [8], Theorem 3.4 and its proof. So now assume $p = 2$. Then again we may use the proof of [5], Theorem 3.4 to deal with A_1, $2A_1$ and $(3A_1)''$. Note that in these cases

$$x_{-\beta_1}(1) \in A_1 \cap Bs_{\beta_1}B \cap U^- \quad ,$$

$$x_{-\beta_1}(1)x_{-\beta_2}(1) \in 2A_1 \cap Bs_{\beta_1}s_{\beta_2}B \cap U^- \quad ,$$

$$x_{-\beta_1}(1)x_{-\beta_2}(1)x_{-\alpha_7}(1) \in (3A_1)'' \cap Bs_{\beta_1}s_{\beta_2}s_{\alpha_7}B \cap U^-$$

To deal with $(3A_1)'$ and $4A_1$, again we may apply the arguments in [5], Theorem 13, since we need results for D_n when $p = 2$ which we proved in §3.1.3. However, it is also possible to show directly that $n_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\alpha_3} \in s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\alpha_3}B \cap (3A_1)'$. To deal with $4A_1$ one can observe that $x = n_{\beta_1} \cdots n_{\beta_7}$ is an involution in Bw_0B. Then $\dim \mathcal{O}_x \geq \ell(w_0) + \operatorname{rk}(1 - w_0) = 70$ by Theorem 3.2, so that $x \in 4A_1$ by Table 5. We have proved

Proposition 3.14 *Let $\mathcal{O}$ be a non-trivial unipotent conjugacy class in E_7. Then $\mathcal{O}$ is spherical if and only if $\mathcal{O} = A_1, 2A_1$, $(3A_1)'$, $(3A_1)''$ or $4A_1$. In each case $\mathcal{O}$ satisfies $(*)$. If $\operatorname{char} k = 2$, these are precisely the classes consisting of involutions.*

3.2.3 Type E_8.

We put

$$\beta_1 = (2,3,4,6,5,4,3,2),\ \beta_2 = (2,2,3,4,3,2,1,0),\ \beta_3 = (0,1,1,2,2,2,1,0),$$
$$\beta_4 = (0,1,1,2,1,0,0,0),\ \beta_5 = \alpha_7,\ \beta_6 = \alpha_5,\ \beta_7 = \alpha_3,\ \beta_8 = \alpha_2.$$

For groups of type E_8 we have to consider $p = 2, 3, 5$. If $p = 3$ or 5, then we may apply the arguments in [5], Theorem 13 to prove that the orbits of type A_1, $2A_1$, $3A_1$ and $4A_1$ are spherical, since to handle $3A_1$ and $4A_1$ we need results for D_4 and D_6 which holds due to [8], Theorem 3.4 and its proof. So now assume $p = 2$. Then again we may use the proof of [5], Theorem 3.4 to deal with A_1 and $2A_1$. Note that in these cases

$$x_{-\beta_1}(1) \in A_1 \cap Bs_{\beta_1}B \cap U^- \quad ,$$

$$x_{-\beta_1}(1)x_{-\beta_2}(1) \in 2A_1 \cap Bs_{\beta_1}s_{\beta_2}B \cap U^- \quad ,$$

To deal with $3A_1$ and $4A_1$, again we may apply the arguments in [5], Theorem 13, since we need results for D_n when $p = 2$ which we proved in §3.1.3. However, it is also possible to show directly that $n_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\beta_5} \in s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5}B \cap 3A_1$. To deal with $4A_1$ one can observe that $x = n_{\beta_1} \cdots n_{\beta_8}$ is an involution in Bw_0B. Then $\dim \mathcal{O}_x \geq \ell(w_0) + \mathrm{rk}(1 - w_0) = 128$ by Theorem 3.2, so that $x \in 4A_1$ by Table 5. We have proved

Proposition 3.15 *Let $\mathcal{O}$ be a non-trivial unipotent conjugacy class in E_8. Then $\mathcal{O}$ is spherical if and only if $\mathcal{O} = A_1, 2A_1$, $3A_1$, or $4A_1$. In each case $\mathcal{O}$ satisfies $(*)$. If $\mathrm{char}\, k = 2$, these are precisely the classes consisting of involutions.*

3.2.4 Type F_4.

We put

$$\begin{aligned} \beta_1 &= (2,3,4,2), \quad \beta_2 = (0,1,2,2), \\ \beta_3 &= (0,1,2,0), \quad \beta_4 = (0,1,0,0) \end{aligned}$$

also γ_1 is the highest short root $(1,2,3,2)$.

For groups of type F_4 we have to consider $p = 2, 3$. If $p = 3$, then we may apply the arguments in [5], Theorem 13 to prove that the orbits of type A_1, $\tilde{A}_1$, $A_1\tilde{A}_1$ and are spherical, since to handle $A_1\tilde{A}_1$ we need results for D_4 which holds due to [8], Theorem 3.4 and its proof. So now assume $p = 2$. Here there are more conjugacy classes $\mathcal{O}_u$ (due to the presence of the graph automorphism of G) for which $\dim \mathcal{O}_u \leq \dim B$ (see Table 5). Each class consists of involutions. We may take the following representatives

$$x_{-\beta_1}(1) \in A_1 \cap Bs_{\beta_1}B \cap U^- \quad ,$$

$$x_{-\gamma_1}(1) \in \tilde{A}_1 \cap Bs_{\gamma_1}B \cap U^- \quad ,$$

To deal with $\tilde{A}_1^{(2)}$, we observe that $u = x_{\beta_1}(1)x_{\gamma_1}(1) \in \tilde{A}_1^{(2)}$ by [2], (13.1). Let K be the C_2-subgroup of G with basis $\{(1,1,1,0), \beta_2\}$. Then β_1 and γ_1 are the highest long and short root in K respectively. A direct calculation in C_2 shows that u is conjugate to $v = x_{\beta_1}(1)x_{\beta_2}(1)$, hence

$$x_{-\beta_1}(1)x_{-\beta_2}(1) \in \tilde{A}_1^{(2)} \cap Bs_{\beta_1}s_{\beta_2}B \cap U^-$$

Finally, to deal with $A_1\tilde{A}_1$ we observe that $x = n_{\beta_1} \cdots n_{\beta_4}$ is an involution in Bw_0B. Then $\dim \mathcal{O}_x \geq \ell(w_0) + \mathrm{rk}(1 - w_0) = 28$ by Theorem 3.2, so that $x \in A_1\tilde{A}_1$ by Table 5.

$\mathcal{O}$	J	$w(\mathcal{O})$	$\dim \mathcal{O}$
A_1	$\{2,3,4\}$	s_{β_1}	16
$\tilde{A}_1$	$\{2,3\}$	$s_{\beta_1}s_{\beta_2}$	22
$A_1\tilde{A}_1$	$\emptyset$	w_0	28

Table 6: F_4, $p = 3$ (or any $\mathrm{char}\ k \neq 2$).

$\mathcal{O}$	J	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
A_1	$\{2,3,4\}$	s_{β_1}	n_{β_1}	16
$\tilde{A}_1$	$\{1,2,3\}$	s_{γ_1}	n_{γ_1}	16
$\tilde{A}_1^{(2)}$	$\{2,3\}$	$s_{\beta_1}s_{\beta_2}$	$n_{\beta_1}n_{\beta_2}$	22
$A_1\tilde{A}_1$	$\emptyset$	w_0	$n_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\beta_4}$	28

Table 7: F_4, $\mathrm{char}\ k = 2$.

Proposition 3.16 *Let $\mathcal{O}$ be a non-trivial unipotent conjugacy class in F_4. Then $\mathcal{O}$ is spherical if and only if it is listed in Table 5. In each case $\mathcal{O}$ satisfies $(*)$. If $\mathrm{char}\ k = 2$, these are precisely the classes consisting of involutions.*

We note that in $G_{\mathbb{C}}$ there is an involution σ such that $C(\sigma)$ is of type B_4 and such that $\mathcal{O}_\sigma$ lies over s_{γ_1}. We also observe that if $p = 2$, then $x_{\alpha_4}(1) \in A_1$, $x_{\alpha_1}(1) \in \tilde{A}_1$, $x_{\alpha_1}(1)x_{\alpha_3}(1) \in A_1\tilde{A}_1$ and these exhaust the conjugacy classes of involutions with representative of the form $\prod_{i \in K} x_{\alpha_i}(1)$, $K \subseteq \Pi$. In particular $\tilde{A}_1^{(2)}$ has no representative of the form $\prod_{i \in K} x_{\alpha_i}(1)$.

3.2.5 Type G_2.

We put $\beta_1 = (3,2),\ \beta_2 = \alpha_1$, $\gamma_1 = (2,1)$ (the highest short root).

For groups of type G_2 we have to consider $p = 2$, 3. The p.o. set of unipotent conjugacy classes is described in the tables in [29], Proposition II 10.4.

If $p = 2$, the classification of unipotent conjugacy classes $\mathcal{O}$ for which $\dim\mathcal{O} \leq \dim B$ is the same as over $\mathbb{C}$ and each class consists of involutions. We may take

$$x_{-\beta_1}(1) \in A_1 \cap Bs_{\beta_1}B \cap U^- \quad ,$$

To deal with $\tilde{A}_1$ we observe that $x = n_{\beta_1}n_{\beta_2}$ is an involution in Bw_0B. Then $\dim\mathcal{O}_x \geq \ell(w_0) + \mathrm{rk}(1-w_0) = 8$ by Theorem 3.2, so that $x \in \tilde{A}_1$ by Table 5.

$\mathcal{O}$	J	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim\mathcal{O}$
A_1	$\{1\}$	s_{β_1}	n_{β_1}	6
$\tilde{A}_1$	$\emptyset$	$w_0 = s_{\beta_1}s_{\beta_2}$	$n_{\beta_1}n_{\beta_2}$	8

Table 8: G_2, $p = 2$.

So now assume $p = 3$. Here there are more conjugacy classes $\mathcal{O}$ for which $\dim\mathcal{O}_u \leq \dim B$ (see Table 5), due to the presence of the graph automorphism of G. We may take the following representatives

$$x_{-\beta_1}(1) \in A_1 \cap Bs_{\beta_1}B \cap U^- \quad ,$$

$$x_{-\gamma_1}(1) \in \tilde{A}_1 \cap Bs_{\gamma_1}B \cap U^- \quad ,$$

To deal with $\tilde{A}_1^{(3)}$, we observe that since $A_1 \leq \tilde{A}_1^{(3)}$ and $\tilde{A}_1 \leq \tilde{A}_1^{(3)}$, we get $s_{\beta_1} \leq w(\tilde{A}_1^{(3)})$ and $s_{\gamma_1} \leq w(\tilde{A}_1^{(3)})$, so that $w(\tilde{A}_1^{(3)}) = w_0$, and we are done since $\dim\tilde{A}_1^{(3)} = 8$.

$\mathcal{O}$	J	$w(\mathcal{O})$	$\dim\mathcal{O}$
A_1	$\{1\}$	s_{β_1}	6
$\tilde{A}_1$	$\{2\}$	s_{γ_1}	6
$\tilde{A}_1^{(3)}$	$\emptyset$	$s_{\beta_1}s_{\beta_2}$	8

Table 9: G_2, $p = 3$.

Proposition 3.17 *Let $\mathcal{O}$ be a non-trivial unipotent conjugacy class in G_2. Then $\mathcal{O}$ is spherical if and only if it is listed in Table 5. In each case $\mathcal{O}$ satisfies $(*)$. If $\operatorname{char} k = 2$, these are precisely the classes consisting of involutions.* □

Note that if $p = 3$, then $x_{\alpha_2}(1) \in A_1$, $x_{\alpha_1}(1) \in \tilde{A}_1$, while $\tilde{A}_1^{(3)}$ has no representative of the form $\prod_{i\in K} x_{\alpha_i}(1)$, $K \subseteq \Pi$.

This completes the classification of spherical unipotent conjugacy classes in bad characteristic. In particular we have proved that

Theorem 3.18 *Let $\mathcal{O}$ be a non-trivial unipotent conjugacy class of a simple algebraic group in characteristic 2. Then $\mathcal{O}$ is spherical if and only if it consists of involutions.* □

This clearly holds for every connected reductive algebraic group in characteristic 2.

Remark 3.19 In each case there exists a unique maximal spherical conjugacy class $\mathcal{O}_{\max}$, and $w(\mathcal{O}_{\max}) = w_0$. The union $\mathcal{U}^{\rm sph}$ of all spherical unipotent orbits is in the closure of $\mathcal{O}_{\max}$.

4 Symmetric homogeneous spaces

In this section we shall prove that if G is a connected reductive algebraic group over the algebraically close field k of characteristic 2, then $H = C(\sigma)$ is a spherical subgroup of G for every involutory automorphism σ of G. This was proved by Vust in [33] in characteristic zero (see also [23] over $\mathbb{C}$). Then Springer extended the result to odd characteristic in [30]. In [28], Seitz gives an alternative proof of Springer's result.

We shall use a generalization of Theorem 3.2. Here G is any connected reductive algebraic group over k, any characteristic.

Let τ be an automorphism of G fixing B and T, and consider $G : \langle\tau\rangle$. Assume τ has order r. Then we have the Bruhat decomposition

$$G : \langle\tau\rangle = \bigcup_{w\in W,\ i\in\mathbb{Z}_r} B\tau^i wB$$

Let $\mathcal{O}$ be a G-orbit in $G : \langle\tau\rangle$. Then there exists a unique $i \in \mathbb{Z}_r$ such that $\mathcal{O} \subseteq \bigcup_{w\in W} B\tau^i wB$, and there is a unique $z = z(\mathcal{O})$ such that $\mathcal{O} \cap B\tau^i zB$ is open dense in $\mathcal{O}$. In particular

$$\overline{\mathcal{O}} = \overline{\mathcal{O} \cap B\tau^i zB} \subseteq \overline{B\tau^i zB} = \tau^i\, \overline{BzB}. \tag{4.1}$$

It follows that if y is an element of $\overline{\mathcal{O}}$ and $y \in B\tau^i wB$, then $w \leq z$ in the Chevalley-Bruhat order of W. Let us observe that if $\mathcal{O}$ is spherical and if $B.x$ is the dense B-orbit in $\mathcal{O}$, then $B.x \subseteq B\tau^i zB$.

We still denote by τ the automorphism of $E = X(T) \otimes \mathbb{R}$ induced by τ (i.e. $\tau(\chi)(t) = \chi(\tau^{-1}t\tau)$ for every $\chi \in X = X(T)$, $t \in T$). For every $w \in W$ we put

$$T^{\tau w} = \{t \in T \mid w^{-1}\tau^{-1}t\tau w = t\}$$

We have $\dim T^{\tau w} = n - \mathrm{rk}(1 - \tau w)$.

Theorem 4.1 *Let $\sigma \in G : \langle\tau\rangle$, $\sigma = \tau^i g$, for a certain $g \in G$, $i \in \mathbb{Z}_r$, and let $\mathcal{O} = G.\sigma$. Suppose that $\mathcal{O}$ contains an element $x \in B\tau^i wB$, for a certain $w \in W$, where U_w is τ-invariant. Then*

$$\dim B.x \geq \ell(w) + \mathrm{rk}(1 - \tau^i w).$$

In particular $\dim \mathcal{O} \geq \ell(w) + \mathrm{rk}(1 - \tau^i w)$. If, in addition, $\dim \mathcal{O} \leq \ell(w) + \mathrm{rk}(1 - \tau^i w)$ then $\mathcal{O}$ is spherical, $w = z(\mathcal{O})$ and $B.x$ is the dense B-orbit in $\mathcal{O}$. $\square$

Proof. Without loss of generality, we may assume $x = \tau^i \dot{w} u$, for a certain representative $\dot{w}$ of w in N and $u \in U$. Let us estimate the dimension of the orbit $B_w.x$, where $B_w = TU_w$.

Let $vt \in C_{B_w}(x)$, with $v \in U_w$, $t \in T$. Then

$$\tau^i \dot{w} uvt = vt\tau^i \dot{w} u = \tau^i \tau^{-i} vt\tau^i \dot{w} u = \tau^i \tau^{-i} v\tau^i \tau^{-i} t\tau^i \dot{w} u = \tau^i\, \tau^{-i} v\tau^i\, \dot{w}\, \dot{w}^{-1}\tau^{-i} t\tau^i \dot{w} u$$

so that, by the uniqueness of the decomposition, $v = 1$ since $t^{-i} v\tau^i \in U_w$. Moreover, from $ut = \dot{w}^{-1}\tau^{-i}t\tau^i\dot{w}u$ it follows $t = \dot{w}^{-1}\tau^{-i}t\tau^i\dot{w}$. Therefore $C_{B_w}(x) \leq T^{\tau^i w}$, thus $\dim C_{B^w}(x) \leq \dim T^{\tau^i w} = n - \mathrm{rk}(1 - \tau^i w)$ and

$$\dim B^w.x = \dim B^w - \dim C_{B^w}(x) \geq \ell(w) + n - n + \mathrm{rk}(1 - \tau^i w) = \ell(w) + \mathrm{rk}(1 - \tau^i w).$$

If, in addition, $\ell(w) + \mathrm{rk}(1 - \tau^i w) \geq \dim \mathcal{O}$, then $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1 - \tau^i w)$. In particular $B.x$ is the dense B-orbit in $\mathcal{O}$. $\square$

We observe that the condition $\tau(U_w) = U_w$ is clearly satisfied if $w = s_{r_1} \cdots s_{r_k}$ where $r_1, \ldots, r_k$ are roots fixed by τ, or if $\{r_1, \ldots, r_k\}$ is a τ invariant set of pairwise orthogonal roots.

In the remainder of this section, we assume that the characteristic of k is 2.

We start with a general remark. Let $G = S\,G_1 \cdots G_t$, where S is the connected component of $Z(G)$, and $G_1, \ldots, G_t$ are the simple components of G. Let σ be an involutorial automorphism of G. Then σ fixes S, and induces a permutation ρ of the set $\{1, \ldots, t\}$. If ρ is non-trivial, then it is the product of disjoint cycles of length 2. Suppose one of these cycles is $(1, 2)$. Then σ induces an isomorphism $\varphi : G_1 \to G_2$. Let $B_1 = T_1U_1$ be a Borel subgroup of G_1, where U_1 is the unipotent radical of B_1, and T_1 a maximal torus. Let V_1 be the maximal unipotent subgroup of the Borel subgroup of G_1 opposite to B_1, and let $R = B_1V_1^{\varphi} \leq G_1G_2$. Then $C_{G_1G_2}(\sigma) \cap R$ is finite, so that $C_{G_1G_2}(\sigma)$ is a spherical subgroup of G_1G_2, since $\dim C_{G_1G_2}(\sigma) + \dim B_1V_1^{\varphi} = \dim G_1G_2$. Of course $C_S(\sigma)$ is a spherical subgroup of S, so that to conclude it is enough to assume that G is a (connected) simple algebraic group.

So now assume G is a simple algebraic group. In the previous section we have already shown that $C(\sigma)$ is a spherical sugbroup of G when σ is an inner involution of G. We are therefore left to deal with outer involutions, which exists only in the following cases: A_ℓ, $\ell \geq 2$, D_ℓ, $\ell \geq 4$ and E_6.

To prove that the fixed point subgroup of any outer involution of G is spherical, we shall use the classification of outer involutions of G as in [29] and [2]. We fix the graph automorphism τ (of order 2) of G, and for each G-orbit $\mathcal{O}$ of outer involutions of G we show that there exists $w = s_{\delta_1} \cdots s_{\delta_\ell} \in W$ such that $\mathcal{O} \cap B\tau wB$ is not empty, $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1 - \tau w)$, δ_i's are pairwise orthogonal positive roots and $\{\delta_1, \ldots, \delta_\ell\}$ is τ invariant. By Theorem 4.1, $\mathcal{O}$ is spherical (with $z(\mathcal{O}) = w$). We consider the various cases: if $\sigma = \tau g \in G : \langle\tau\rangle$, $C(\sigma)$ stands for $C_G(\sigma)$. In each case we use the notation introduced in section 3.

4.1 Type A_n, $n = 2m$, $m \geq 1$.

We take $G = SL(2m+1)$. In this case there is only one (class of) outer involution τ, the graph automorphism of $SL(2m+1)$, and $C(\tau) = SO(2m+1)$. Then

$$\dim SL(2m+1)/SO(2m+1) = 2m^2 + 3m$$

which is the dimension of a Borel subgroup of $SL(2m+1)$. We may take

$$x = \tau n_{\beta_1} \cdots n_{\beta_m} \in \tau w_0 B.$$

Then $x^2 = 1$ since $\tau(\beta_k) = \beta_k$ for each k, so that x lies in $\mathcal{O}_\tau$. Since $\tau w_0 = -1$, we get

$$\ell(w_0) + \mathrm{rk}(\tau w_0 - 1) = \dim B$$

and we are done.

Hence

$\mathcal{O}$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap \tau B w(\mathcal{O}) B$
τ	$w_0 = s_{\beta_1} \cdots s_{\beta_m}$	$\tau n_{\beta_1} \cdots n_{\beta_m}$

Table 10: Outer involutions in $SL(2m+1)$, $m \geq 1$.

4.2 Type A_n, $n = 2m-1$, $m \geq 2$.

We take $G = SL(2m)$. In this case there are two (classes of) outer involutions: τ and $\tau x_{\beta_1}(1)$, with $C(\tau) = Sp(2m)$, $C(\tau x_{\beta_1}(1)) = C_{Sp(2m)}(x_{\beta_1}(1))$. We have

$$\dim SL(2m)/Sp(2m) = 2m^2 - m - 1$$

We put $J = \{1, 3, \ldots, n\}$, $w = w_0 w_J$. We have

$$\ell(w) = \ell(w_0) - \ell(w_J) = m(2m-1) - m = 2m^2 - 2m$$

and

$$\mathrm{rk}(\tau w - 1) = \mathrm{rk}(w_J + 1) = m - 1$$

since $\tau w_0 = -1$ and the (simple) roots in J are pairwise orthogonal. Hence

$$\ell(w) + \mathrm{rk}(\tau w - 1) = 2m^2 - m - 1 = \dim SL(2m)/Sp(2m)$$

We are left to exhibit a conjugate $x \in \tau BwB$ of τ. For this purpose we distinguish 2 cases.

Assume m is even, $m = 2r$. Then there are precisely m positive roots $\gamma_1, \ldots, \gamma_m$ for which $w(\gamma) = -\gamma$, namely

$$\gamma_{2i-1} = e_{2i-1} - e_{2m-2i+1} \quad , \quad \gamma_{2i} = e_{2i} - e_{2m+2-2i}$$

for $i = 1, \ldots, r$ and

$$w = s_{\gamma_1} \cdots s_{\gamma_m}$$

We also note that τ exchanges γ_{2i-1} and γ_{2i} for each $i = 1, \ldots, r$. In this case we take

$$x = g\tau g^{-1}$$

where g is the involution $g = x_{-\gamma_1}(1)x_{-\gamma_3}(1)\cdots x_{-\gamma_{m-1}}(1)$. Then

$$x = \tau x_{-\gamma_1}(1)x_{-\gamma_2}(1)x_{-\gamma_3}(1)\cdots x_{-\gamma_{m-1}}(1)x_{-\gamma_m}(1) \in \tau BwB$$

and we are done.

If m is odd, $m-1 = 2r$, then there are precisely $m-1$ positive roots $\gamma_1, \ldots, \gamma_{m-1}$ for which $w(\gamma) = -\gamma$, namely

$$\gamma_{2i-1} = e_{2i-1} - e_{2m-2i+1} \quad , \quad \gamma_{2i} = e_{2i} - e_{2m+2-2i}$$

for $i = 1, \ldots, r$ and

$$w = s_{\gamma_1}\cdots s_{\gamma_{m-1}}$$

We also note that τ exchanges γ_{2i-1} and γ_{2i} for each $i = 1, \ldots, r$. In this case we take

$$x = g\tau g^{-1}$$

where g is the involution $g = x_{-\gamma_1}(1)x_{-\gamma_3}(1)\cdots x_{-\gamma_{m-2}}(1)$. Then

$$x = \tau x_{-\gamma_1}(1)x_{-\gamma_2}(1)x_{-\gamma_3}(1)\cdots x_{-\gamma_{m-2}}(1)x_{-\gamma_{m-1}}(1) \in \tau BwB$$

and we are done.

We finally deal with $\tau x_{\beta_1}(1)$, $H = C(\tau x_{\beta_1}(1)) = C_{Sp(2m)}(x_{\beta_1}(1))$. We have

$$\dim SL(2m)/H = \dim SL(2m)/Sp(2m) + \dim \mathcal{O}' = 2m^2 - m - 1 + 2m = 2m^2 + m - 1$$

where $\mathcal{O}'$ is the $Sp(2m)$-orbit of $x_{\beta_1}(1)$, which has dimension $2m$ (note that $\dim SL(2m)/H = \dim SL(2m)/SO(2m)$, and $SO(2m)$ is the centralizer of an outer involution of $SL(2m)$ if the characteristic is not 2). Therefore $\dim SL(2m)/H$ is the dimension of a Borel subgroup of $SL(2m)$. As in the case when n is even, we take $w = w_0$,

$$x = \tau n_{\beta_1}\cdots n_{\beta_m} \in \tau w_0 B.$$

Then $x^2 = 1$ since $\tau(\beta_k) = \beta_k$ for each k, and for dimensional reasons, x is conjugate to $\tau x_\beta(1)$.

Hence

$\mathcal{O}$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap \tau B w(\mathcal{O}) B$
τ	$s_{\gamma_1} \cdots s_{\gamma_m}$	$\tau x_{-\gamma_1}(1) \cdots x_{-\gamma_m}(1)$
$\tau x_{\beta_1}(1)$	w_0	τn_0

Table 11: Outer involutions in $SL(2m)$, $m \geq 2$, m even.

and

$\mathcal{O}$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap \tau B w(\mathcal{O}) B$
τ	$s_{\gamma_1} \cdots s_{\gamma_{m-1}}$	$\tau x_{-\gamma_1}(1) \cdots x_{-\gamma_{m-1}}(1)$
$\tau x_{\beta_1}(1)$	w_0	τn_0

Table 12: Outer involutions in $SL(2m)$, $m \geq 3$, m odd.

4.3 Type D_n, $n \geq 4$.

To deal with G of type D_n we shall, as usual, consider $G = SO(2n)$. Then the outer involutions of G are obtained by conjugation with involutions of $O(2n)$. Note that if $n = 4$, and if G is adjoint or simply-connected, then there are other outer involutions in Aut G: however, they are conjugate in Aut G.

Let τ be the involution of $O(2n)$ inducing the graph automorphism of $SO(2n)$, i.e. the graph automorphism acting trivially on $\langle X_{\pm\alpha_i} \mid i \in \{1, \ldots, n-2\}\rangle$ and such that $x_{\alpha_{n-1}}(\xi) \leftrightarrow x_{\alpha_n}(\xi)$, $x_{-\alpha_{n-1}}(\xi) \leftrightarrow x_{-\alpha_n}(\xi)$ for $\xi \in k$.

The classes of involutions in $O(2n) \setminus SO(2n)$ correspond to partitions $2^k \oplus 1^{2n-2k}$ for $k = 1, \ldots, n$, odd k, with τ corresponding to $2 \oplus 1^{2n-2}$. Let $\mathcal{O}_k$ be the class corresponding to $2^k \oplus 1^{2n-2k}$. From [29], 2.9 b) we get

$$\dim \mathcal{O}_k = \dim \mathcal{O}_{k,Sp(2n,\mathbb{C})} - 2n + \lambda_1^*$$

where $\lambda_1^* = c_1 + c_2 = (2n-2k) + k = 2n - k$, hence

$$\dim \mathcal{O}_k = k(2n - k + 1) - 2n + 2n - k = k(2n - k)$$

for $k = 1, \ldots, n$, odd k.

Let

$$\mu_1 = e_1 - e_n \quad , \quad \nu_1 = e_1 + e_n \quad , \quad w = s_{\mu_1} s_{\nu_1} \quad .$$

Then

$$\ell(w) + \mathrm{rk}(1 - \tau w) = 2(n-1) + 1 = 2n - 1 = \dim \mathcal{O}_1$$

and $\tau(\mu_1) = \nu_1$. We have

$$n_{\nu_1} \tau n_{\nu_1} = \tau n_{\mu_1} n_{\nu_1}$$

so that

$$\tau n_{\mu_1} n_{\nu_1} \in \mathcal{O}_1 \cap wB$$

To deal with the remaining classes, we put $m = \left[\frac{n}{2}\right]$ and

$$\mu_i = e_{2i-2} - e_{2i-1} \quad , \quad \nu_i = e_{2i-2} + e_{2i-1} \quad , \quad w_i = s_{\mu_1} s_{\nu_1} \cdots s_{\mu_i} s_{\nu_i}$$

for $i = 2, \ldots, m$.

Arguing as above, we can prove that

$$\ell(w) + \mathrm{rk}(1 - \tau w) = \dim \mathcal{O}_{2i-1}$$

and

$$\tau n_{\mu_1} n_{\nu_1} \cdots n_{\mu_i} n_{\nu_i} \in \mathcal{O}_{2i-1} \cap w_i B$$

for $i = 2, \ldots, m$. In fact it is enough to count the number of Jordan blocks of length 2 in $\tau n_{\mu_1} n_{\nu_1} \cdots n_{\mu_i} n_{\nu_i}$: in $\tau n_{\mu_1} n_{\nu_1}$ there is 1, and in $n_{\mu_i} n_{\nu_i}$ there are 2 for each $i = 2, \ldots, m$.

If n is even, then there are $\frac{1}{2}n$ conjugacy classes of outer involutions and we are done. In particular the maximal one is $2^{n-1} \oplus 1^2$ and corresponds to $w = s_{\mu_1} s_{\nu_1} \cdots s_{\mu_m} s_{\nu_m} = w_0 = -1$.

If n is odd, then there are $\frac{1}{2}(n+1)$ conjugacy classes of outer involutions: the maximal one is $\mathcal{O}_n = 2^n$ which is the only one not in the previous list. We have

$$\dim \mathcal{O}_n = n^2$$

Let n_0 be any representative in N of w_0 with n_0 of order 2 and commuting with τ. Then $x = \tau n_0$ is an involution in $\tau w_0 B$. Since

$$\ell(w_0) + \mathrm{rk}(1 - \tau w_0) = n^2 - n + n = n^2$$

by Theorem 4.1, we have $\dim B.x \geq n^2$. But x is an involution in $O(2n) \setminus SO(2n)$, so that $\dim G.x \leq \dim \mathcal{O}_n = n^2$. Therefore x lies in $\mathcal{O}_n \cap \tau w_0 B$ and we are done.

$\mathcal{O}$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap \tau B w(\mathcal{O}) B$	$\dim \mathcal{O}$
$2^{2i-1} \oplus 1^{2n-4i+2}$ $i = 1, \ldots, m$	$s_{\mu_1} s_{\nu_1} \cdots s_{\mu_i} s_{\nu_i}$	$\tau n_{\mu_1} n_{\nu_1} \cdots n_{\mu_i} n_{\nu_i}$	$(2i-1)(2n-2i+1)$

Table 13: Outer involutions in D_n, $n \geq 4$, $n = 2m$.

$\mathcal{O}$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap \tau B w(\mathcal{O}) B$	$\dim \mathcal{O}$
$2^{2i-1} \oplus 1^{2n-4i+2}$ $i = 1, \ldots, m$	$s_{\mu_1} s_{\nu_1} \cdots s_{\mu_i} s_{\nu_i}$	$\tau n_{\mu_1} n_{\nu_1} \cdots n_{\mu_i} n_{\nu_i}$	$(2i-1)(2n-2i+1)$
2^n	w_0	τn_0	n^2

Table 14: Outer involutions in D_n, $n \geq 4$, $n = 2m+1$.

4.4 Type E_6.

There are two (classes of) outer involutions: τ and $\tau x_{\beta_1}(1)$, where τ is the graph automorphism of G. We recall from §4.4 that

$$\beta_1 = (1,2,2,3,2,1), \quad \beta_2 = (1,0,1,1,1,1)$$
$$\beta_3 = (0,0,1,1,1,0), \quad \beta_4 = (0,0,0,1,0,0)$$

Note that each β_i is fixed by τ.

Let us start with $\tau x_{\beta_1}(1)$. We have $K = C(\tau x_{\beta_1}(1)) \cong C_{F_4}(x_{\beta_1}(1))$, $\dim K = 36$. Let $x = \tau n_{\beta_1} n_{\beta_2} n_{\beta_3} n_{\beta_4}$. Since x is an involution in $\tau w_0 B$, with $\ell(w_0) + \mathrm{rk}(\tau w_0 - 1) = 36 + 6 = \dim E_6/K$, it follows that $\tau x_{\beta_1}(1) \sim x$.

To deal with τ, we put $\delta_1 = (1,1,2,2,1,1)$, $\delta_2 = (1,1,1,2,2,1)$. We have

$$\dim E_6/F_4 = 26$$

Note that $\tau(\delta_1) = \delta_2$ and

$$\ell(s_{\delta_1} s_{\delta_2}) + \mathrm{rk}(\tau s_{\delta_1} s_{\delta_2} - 1) = 24 + 2 = 26$$

In fact here $J = \{2,3,4,5\}$, $w = s_{\delta_1} s_{\delta_2} = w_0 w_J$.

We show that $\tau \sim \tau n_{\delta_1} n_{\delta_2}$. Let $g = x_{-\delta_1}(1)$. Then

$$g\tau g^{-1} = \tau x_{-\delta_1}(1) x_{-\delta_2}(1) \in \tau BwB$$

Moreover, since $[\tau, x_{-\delta_1}(1)x_{-\delta_2}(1)] = 1$, we get

$$x_{\delta_1}(1)x_{\delta_2}(1)\tau x_{-\delta_1}(1)x_{-\delta_2}(1)x_{\delta_1}(1)x_{\delta_2}(1) = \tau n_{\delta_1} n_{\delta_2}$$

and we are done. We summarize in

$\mathcal{O}$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap \tau Bw(\mathcal{O})B$
τ	$s_{\delta_1} s_{\delta_2}$	$\tau n_{\delta_1} n_{\delta_2}$
$\tau x_{\beta_1}(1)$	w_0	$\tau n_{\beta_1} n_{\beta_2} n_{\beta_3} n_{\beta_4}$

Table 15: Outer involutions in E_6.

This completes the list of outer involutions of simple algebraic groups in characteristic 2. We have proved that

Theorem 4.2 *Let G be a reductive connected algebraic group in characteristic 2, and let σ be any involutory automorphism of G. Then the fixed point subgroup H of σ is a spherical subgroup of G.* $\square$

We conclude with another application of Theorem 4.1.

4.5 Type G_2 in D_4.

We show briefly how one can prove that the subgroup of type G_2 in D_4 is spherical (in all characteristics). Let us assume G of type D_4. Without loss of generality, we may assume G is adjoint. Hence if we denote by φ the graph automorphism of G fixing α_2 and mapping $\alpha_1 \mapsto \alpha_3 \mapsto \alpha_4 \mapsto \alpha_1$, then the fixed point subgroup K of φ is of type G_2. Let $\delta_1 = \alpha_1+\alpha_2+\alpha_3$, $\delta_2 = \alpha_1 + \alpha_2 + \alpha_4$, $\delta_3 = \alpha_2 + \alpha_3 + \alpha_4$, and let $w = w_0 s_2 = s_{\delta_1} s_{\delta_2} s_{\delta_3}$. We have

$$\ell(w) + \mathrm{rk}(1 - \varphi w) = 14 = \dim D_4/G_2 \quad .$$

It remains to show that a G-conjugate of φ lies in $\varphi B w_0 s_2 B \subseteq G : \langle\varphi\rangle$.

Let $g = x_{-\delta_1}(\xi_1)x_{-\delta_2}(\xi_2)x_{-\delta_3}(\xi_3)$. Then

$$g\varphi g^{-1} = \varphi x_{-\delta_1}(-\xi_1 - \xi_3)x_{-\delta_2}(-\xi_2 - \xi_1)x_{-\delta_3}(-\xi_3 + \xi_2)$$

If we choose ξ_1, ξ_2, ξ_3 such that $\xi_1 + \xi_3$, $\xi_2 + \xi_1$ and $-\xi_3 + \xi_2$ are non-zero, then $g\varphi g^{-1} \in \varphi B w_0 s_2 B$ and we are done.

Remark 4.3 If the characteristic is zero, we may apply the arguments in [11]. Since $T^{\varphi w} = H_{\alpha_2}$ is connected, it follows that, in the simply-connected case, the monoid $\lambda(D_4/G_2)$ of B-weights in $k[D_4/G_2]$ is generated by $\omega_1, \omega_3, \omega_4$, since G_2 is connected and it has no non-trivial characters, so that the monoid $\lambda(D_4/G_2)$ is free (and it contains $(1-\varphi w)P^+$ which is the monoid generated by $\omega_1 + \omega_3$, $\omega_1 + \omega_4$, $\omega_3 + \omega_4$, $\omega_1 + \omega_3 + \omega_4$) (see also [20]).

References

[1] A. V. ALEKSEEVSKII, *Component groups of centralizers of unipotent elements in semisimple algebraic groups,* Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk. Gruzin SSR 62, 5–27 (1979), Lie groups and invariant theory, Amer. Math. Soc. Transl. Ser. 2, 213, Amer. Math. Soc. Providence, RI (2005).

[2] M. ASCHBACHER, G.M. SEITZ *Involutions in Chevalley groups over fields of even order,* Nagoya Math. J., 63, 1–91 (1976).

[3] N. BOURBAKI, *Éléments de Mathématique. Groupes et Algèbres de Lie, Chapitres 4,5, et 6,* Masson, Paris (1981).

[4] J. BRUNDAN, *Dense orbits and double cosets,* Algebraic groups and their representations (Cambridge, 1997), 259–274, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 517, Kluwer Acad. Publ., Dordrecht, (1998).

[5] N. CANTARINI, G. CARNOVALE, M. COSTANTINI, *Spherical orbits and representations of $\mathcal{U}_\varepsilon(\mathfrak{g})$,* Transformation Groups, 10, No. 1, 29–62 (2005).

[6] G. CARNOVALE, *Spherical conjugacy classes and involutions in the Weyl group,* Math. Z. 260(1), 1–23 (2008)

[7] G. CARNOVALE, *Spherical conjugacy classes and Bruhat decomposition,* preprint ArXiv:0808.1818v2 (to appear in Ann. Inst. Fourier, Grenoble).

[8] G. CARNOVALE, *A classification of spherical conjugacy classes in good characteristic,* preprint ArXiv:0811.2641v2 (to appear in Pacific J. Math.).

[9] M. COSTANTINI *On the lattice automorphisms of certain simple algebraic groups,* Rend. Sem. Mat. Univ. Padova 90, 141–157 (1993).

[10] M. COSTANTINI *The lattice automorphisms of simple algebraic groups over $\overline{\mathbb{F}}_2$,* Manuscripta Math. 91, 1–16 (1996).

[11] M. COSTANTINI, *On the coordinate ring of spherical conjugacy classes,* Math. Z. on line (2009)

[12] R. W. CARTER, *Simple Groups of Lie Type,* John Wiley (1989).

[13] R. W. CARTER, *Finite Groups of Lie Type,* John Wiley (1985).

[14] K.Y. CHAN, J.-H. LU, S.K.M. TO, *On intersections of conjugacy classes and Bruhat cells,* preprint ArXiv:0906.2254v1.

[15] R. FOWLER, G. RÖHRLE *Spherical nilpotent orbits in positive characteristic,* Pacific J. Math. 237, 241–286 (2008).

[16] W.H. HESSELINK, *Nilpotency in classical groups over a filed of characteristic 2,* Math. Z. 54, 165–181 (1979).

[17] J.E. HUMPHREYS, *Linear Algebraic Groups,* Springer-Verlag, New York (1995).

[18] N. IWAHORI, *Centralizers of involutions in finite Chevalley groups,* In: “Seminar on algebraic groups and related finite groups”. LNM 131, 267–295, Springer-Verlag, Berlin Heidelberg New York (1970).

[19] F. KNOP, *On the set of orbits for a Borel subgroup,* Comment. Math. Helv. 70, 285–309 (1995).

[20] M. KRÄMER, *Sphärische Untergruppen in kompakten zusammenhangenden Liegruppen,* Compositio Math. **38** (1979), 129–153.

[21] R. LAWTHER, M.W. LIEBECK, G.M. SEITZ *Fixed point spaces in actions of exceptional algebraic groups,* Pacific J. Math. 205, 339–391 (2002).

[22] G. LUSZTIG, *Remarks on Springer's representations,* preprint ArXiv:0811.0370v1 (3 Nov 2008).

[23] T. MATSUKI, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups,* J. Math. Soc. Japan 31, 331–357 (1979).

[24] D. PANYUSHEV, *Complexity and rank of homogeneous spaces,* Geom. Dedicata, 34, 249–269 (1990).

[25] D. PANYUSHEV, *Complexity and nilpotent orbits,* Manuscripta Math. 83, 223–237 (1994).

[26] D. PANYUSHEV, *On spherical nilpotent orbits and beyond,* Ann. Inst. Fourier, Grenoble 49(5), 1453–1476 (1999).

[27] D. PANYUSHEV, *Some amazing properties of spherical nilpotent orbits,* Math. Z., 245, 557–580 (2003).

[28] G.M.. SEITZ, *Double cosets in algebraic groups,* Algebraic groups and their representations (Cambridge, 1997), 214–257, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 517, Kluwer Acad. Publ., Dordrecht, (1998).

[29] J.N. SPALTENSTEIN, *Classes unipotentes et sous-groupes de Borel,* LNM 946, Springer-Verlag, Berlin Heidelberg New York (1982).

[30] T.A. SPRINGER, *Some results on algebraic groups with involutions,* Algebraic groups and related topics (Kyoto/Nagoya, 1983), 525–543, Adv. Stud. Pure Math. 6, North-Holland, Amsterdam, (1985).

[31] T.A. SPRINGER, *Linear Algebraic Groups*, Second Edition, Progress in Mathematics 9, Birkhäuser (1998).

[32] R. STEINBERG, *Lectures on Chevalley groups,* Yale University (1967).

[33] T. VUST, *Opération de groupes réductifs dans un type de cônes presques homogènes,* Bull. Soc. Math. France 102, 317–333 (1974).

[34] G.E. WALL, *On the conjugacy classes in the unitary, symplectic and orthogonal groups,* J. Austral. Math. Soc. 3, 62–89 (1963).